Real Option Pricing

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Real Options

A **real option** is an economically valuable right to make or else abandon some choice that is available to the managers of a company, often concerning business projects or investment opportunities.
Real options can include the decision to expand, defer or wait, or abandon a project entirely.

Real Options




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Real options are most appropriate when the economic environment and market conditions relating to a particular project are highly volatile yet flexible.
Stable or rigid environments will not benefit much from real options value analysis and should use more traditional corporate finance techniques instead.

Problem Statement

In corporate finance, firms may be faced with the situation to determine the optimal investment time when they launch new projects.

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We introduce the optimal investment decision problem on the finite time horizon, where the instantaneous cash flow process of the firm follows the regime-switching jump-diffusion (RSJD) model.

State of a Economy $(R_t)_{t\in[0,T_p]}$

A regime-switching Markov process is involved in order to deal with multiple regimes in the business cycle.

A continuous-time Markov process $(R_t)_{t\in[0,T_p]}$ on a filtered probability space $(\Omega, F, (F_t)_{t\in[0,T_p]}, \mathbb{P})$ denotes a state of the economy in a finite state space $\mathcal{M} = \{e_1, e_2, \cdots, e_K\}$, where T_p is an expiration date of a new project of the firm and a vector e_i is a unit vector with 1 in the i-th entry and zeros elsewhere in a K-dimensional space.

Then the Markov chain process R_t has a semimartingale representation of the form

$$R_t = R_0 + \int_0^t AR_{s-}ds + M_t.$$

State of a Economy $(R_t)_{t \in [0, T_p]}$

The term $\sigma_t = \langle \sigma, R_t \rangle$ represents the volatility in the regime-switching jump diffusion model, which varies according to the current state of economic, R_t.

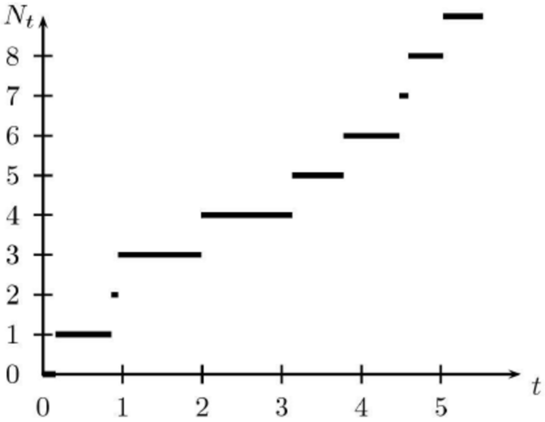
Cash Flow Process $(X_t)_{t\in[0,T_p]}$

We assume that an instantaneous cash flow process of the firm $(X_t)_{t\in[0,T_p]}$ follows a regime-switching jump-diffusion (RSJD) model after the firm undertakes the irreversible investment with a sunk cost.

Regime-Switching Jump Diffusion model

In the Regime-Switching Jump Diffusion (RSJD) model, the **Wiener process** models the continuous, stochastic fluctuations of asset prices, while the **Poisson process** captures the discrete, sudden jumps due to unexpected events or shocks such as regulatory changes, geopolitical tensions and natural disasters.

Poisson Process



Cash Flow Process $(X_t)_{t\in[0,T_p]}$

The stochastic differential equation of X_t is given by

$$\frac{dX_t}{X_{t-}} = (\mu_t - \lambda_t\zeta_t)dt + \sigma_t dW_t + \eta_t dN_t$$

where $\mu_t = \langle \mu, R_t \rangle$ is an expected rate of return with $\mu = (\mu_1, \mu_2, \cdots, \mu_K)^T$, $\sigma_t = \langle \sigma, R_t \rangle$ is a volatility of the cash flow process X_t with $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_K)^T$ satisfying a condition $\sigma_i > 0$ for $1 \leq i \leq K$, W_t is a Wiener process, $N_t = \langle \mathbf{N}_t, R_t \rangle$ with $\mathbf{N}_t = (N_t^1, N_t^2, \cdots, N_t^K)^T$ is a Poisson process having an intensity $\lambda_t = \langle \lambda, R_t \rangle$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_K)^T$, $\eta = \langle \eta, R_t \rangle$ is a stochastic process to generate jump sizes from X_{t-} to X_t with a random variable $\eta = (\eta_1, \eta_2, \cdots, \eta_K)^T$ satsfying a condition $\eta_i > -1$ for $1 \leq i \leq K$, and $\zeta_t = \langle \zeta, R_t \rangle$ is the inner product of ζ and R_t with $\zeta = \mathbb{E}[\eta]$. The stochastic processes $W_t, R_t, N_t^1, \cdots, N_t^K$, and $\eta_1, \cdots, \eta_K$ are mutually independent and the jumps of two processes R_t and N_t do not occur simultaneously almost surely.

Cash Flow Process $(X_t)_{t\in[0,T_p]}$

The SDE of X_t can be decomposed into two parts: the diffusion component and the jump component.

The diffusion component is written as

$$\frac{dX_t}{X_{t-}} = \mu_t dt + \sigma_t dW_t.$$

The jump component is written as

$$\frac{dX_t}{X_{t-}} = -\lambda_t \zeta_t dt + \eta_t dN_t$$

where $N_t = \langle \mathbf{N}_t, R_t\rangle$ with $\mathbf{N}_t = (N_t^1, N_t^2, \cdots, N_t^K)^T$ is a Poisson process having an intensity $\lambda_t = \langle \lambda, R_t\rangle$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_K)^T$, $\eta = \langle \eta, R_t\rangle$ is a stochastic process to generate jump sizes from X_{t-} to X_t with a random variable $\eta = (\eta_1, \eta_2, \cdots, \eta_K)^T$ satsfying a condition $\eta_i > -1$ for $1 \leq i \leq K$, and $\zeta_t = \langle \zeta, R_t\rangle$ is the inner product of ζ and R_t with $\zeta = \mathbb{E}[\eta]$.

$Q(t, x, e_i)$

Then the current value of the future cash flow at time t is given by

$$Q(t, x, e_i) = \mathbb{E}\left[\int_t^{T_p} e^{-\kappa(s-t)} X_s ds \mid X_t = x, R_t = e_i\right],$$

where $\kappa_t = \langle \kappa, R_t \rangle$ is a discount rate with $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_K)^T$.

Objective $J(t, x, e_i)$

The objective of the firm is to find an optimal investment time to maximize the expected discounted payoff on the finite time horizon

$$J(t, x, e_i) = \sup_{\tau \in [t, T_r]} \mathbb{E}\left[e^{-\kappa(\tau - t)} \max(Q(\tau, X_\tau, R_\tau) - I, 0) \mid X_t = x, R_t = e_i \right],$$

where T_r is an expiration date of the option to invest in the project and $[t, T_r]$ is a set of all stopping times τ with respect to the filtration $(F_t)_{t \in [0, T_p]}$ satisfying $t \leq \tau \leq T_r$.

Closed-form Solution

The value of the project satisfies the system of the PIDE

$$\begin{aligned}\partial Q_i \partial t(t,x) \quad & +\tfrac{1}{2}\sigma_i^2 x^2 \frac{\partial^2 Q_i}{\partial x^2}(t,x) + (\mu_i, \lambda_i \zeta_i) x \frac{\partial Q_i}{\partial x}(t,x) - (\kappa_i + \lambda_i) Q_i(t,x) + x \\ & +\lambda_i \int_0^\infty Q_i(t,xz) g_i(z) dz + \sum_{j=1, j\neq i}^{K} \gamma_{ji} \left(Q_j(t,x) - Q_i(t,x)\right) = 0\end{aligned}$$

and the terminal condition $Q_i(T_p, x) = 0$ with $Q_i(t,x) = Q(t,x,e_i)$ for all $(t,x,e_i) \in (0, T_p] \times (0,\infty) \times \mathcal{M}$, where $g_i(z) = g(z,e_i)$ is the probability density function of the jump sizes $\eta_i + 1$ at the i-state of the economy.

Closed-form Solution

We use the linear complementarity problem (LCP) and apply the 2-step backward differentiation formula (BDF2 method) combined with the operator splitting method.