Solving Merton's Portfolio Problem

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Merton's Portfolio Problem

Problem Statement

- Balance constraint implies the following process for Wealth W_t

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t.$$

Merton's Portfolio Problem

The goal of Merton's portfolio problem is to maximize the lifetime-aggregated utility of consumption by selecting the optimal allocation and consumption at each time point.

Optimal Value Function

- At any time t, determine optimal $[\pi(t, W_t), c(t, W_t)]$ to maximize:

$$\mathbb{E}_t \left[\int_t^T \frac{e^{-\rho(s-t)} \cdot c_s^{1-\gamma}}{1-\gamma} ds + \frac{e^{-\rho(T-t)} \cdot \epsilon^{\gamma} \cdot W_T^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

where $\rho \geq 0$ is the utility discount rate.

Continuous-Time Stochastic Control

- The State at time t is (t, W_t)
- The Action at time t is $[\pi_t, c_t]$
- The Reward per unit time at time t is $U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$
- The Return at time t is the accumulated discounted Reward:

$$\int_t^T e^{-\rho(s-t)} \cdot \frac{c_s^{1-\gamma}}{1-\gamma} \cdot ds$$

- Find Policy: $(t, W_t) \rightarrow [\pi_t, c_t]$ that maximizes the Expected Return

Optimal Allocation and Consumption

Finally, we get:

$$\pi^*(t, W_t) = \frac{\mu - r}{\sigma^2 \gamma}$$

$$c^*(t, W_t) = \frac{W_t}{f(t)} = \begin{cases} \frac{\nu \cdot W_t}{1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)}} & \text{for } \nu \neq 0 \\ \frac{W_t}{T-t+\epsilon} & \text{for } \nu = 0 \end{cases}$$

$$V^*(t, W_t) = \begin{cases} \frac{\left(1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)}\right)^\gamma}{\nu^\gamma} \cdot \frac{W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu \neq 0 \\ \frac{(T-t+\epsilon)^\gamma \cdot W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu = 0 \end{cases}$$

Optimal Allocation and Consumption

- With Optimal Allocation Consumption, the Wealth process is:

$$\frac{dW_t}{W_t} = \left(r + \frac{(\mu - r)^2}{\sigma^2\gamma} - \frac{1}{f(t)}\right) \cdot dt + \frac{\mu - r}{\sigma\gamma} \cdot dz_t$$

Domain

$(x, t) \in [W_{min}, W_{max}] \times [0, T_{max}]$

Structure

We use 3 neural networks to predict

- Fraction of wealth allocated to risky asset, $\pi(t, W_t)$,
- Wealth consumption per unit time, $c(t, W_t)$,
- Value function, $V(t, W_t)$

Structure of RiskNN

- Args: State (t, W_t)
- 4 hidden layers with 100 perceptrons for each layers
- Softplus as activation functions
- Returns: Prediction of fraction of wealth allocated to risky asset, $\pi(t, W_t)$

Structure of ConsumpNN

- Args: State (t, W_t)
- 4 hidden layers with 100 perceptrons for each layers
- LeakyReLU and Softplus as activation functions
- Returns: Prediction of wealth consumption per unit time, $c(t, W_t)$

Structure of ValueNN

- Args: State (t, W_t)
- 4 hidden layers with 100 perceptrons for each layers
- LeakyReLU and Softplus as activation functions
- Returns: Prediction of Value function, $V(t, W_t)$

Optimizer

Optimizer: Adam with learning rate 1e-4

Learning

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t \qquad (2)$$

We rewrite (2) as follows:

$\mathrm{dW}_t = -c_t \cdot dt$
$\mathrm{dW}_t = (\pi_t \cdot (\mu - r) + r) \cdot W_t \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$

Results for Value

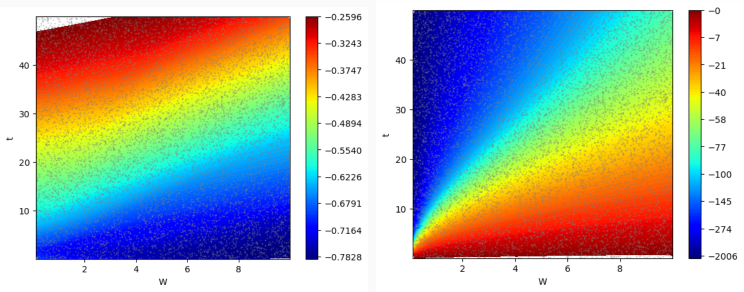

Figure 1: (Left) Value predicted by the ValueNN (Right) The exact Value

Results for Value

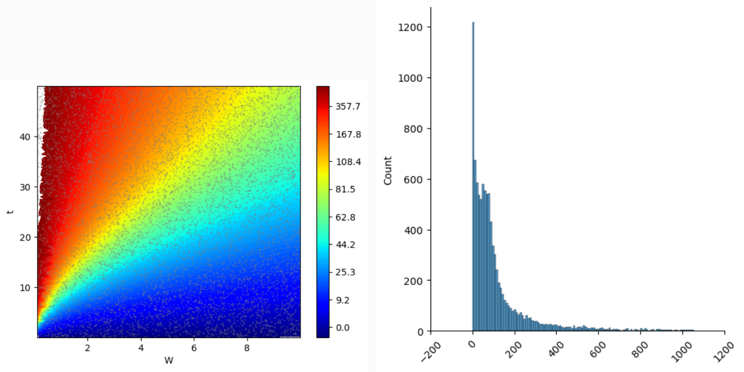

Figure 2: Error for Value

Results for Consumption

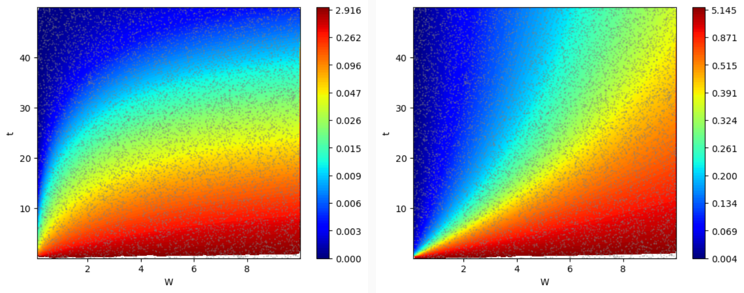

Figure 3: (Left) Consumption predicted by ConsumpNN $\times W_t/(1e-6+t)$ (Right) The exact Consumption value

Results for Consumption

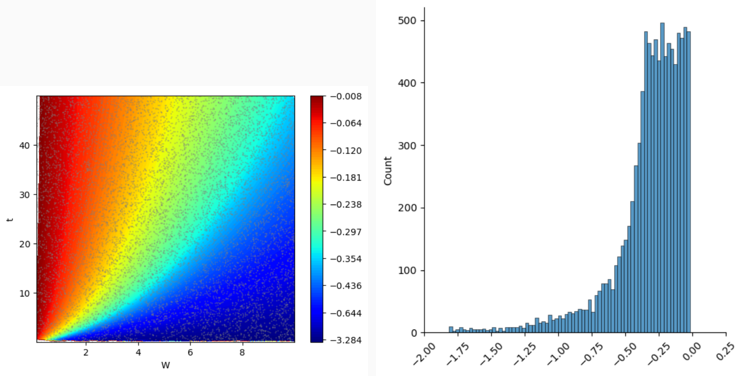

Figure 4: Error for Consumption

Results for Risk

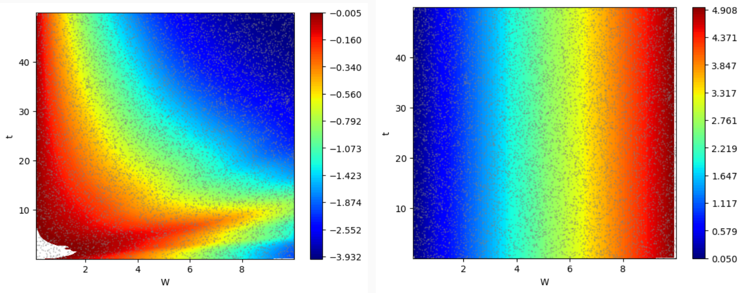

Figure 5: (Left) Risk predicted by RiskNN (Right) The exact Risk value

Results for Risk

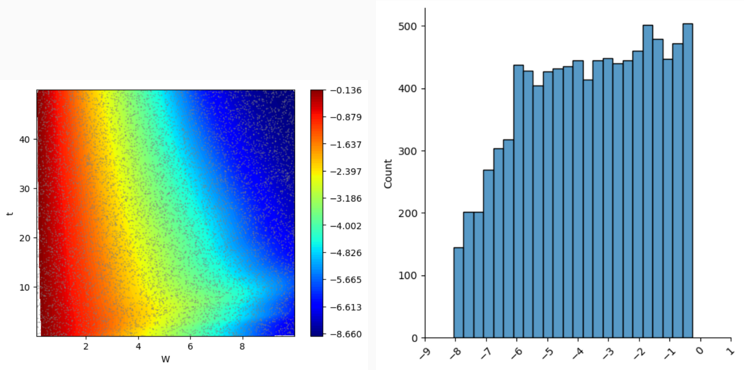

Figure 6: Error for Risk

Discussion

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