A Maximum Principle for SDEs of Mean-Field Type

Sungkyunkwan University
Department of Mathematics
Hyelin choi

Contents

Contents

Stochastic LQ control problem	Zhou, X.Y., Li, D.: Continuous-time mean-variance portfolio selection: a stochastic LQ framework. Appl. Math. Optim. **42**, 19–33 (2000)
Maximum principle	This paper

This paper's Maximum Principle is the stochastic, mean-field extension of Pontryagin's Maximum Principle.

I will briefly review the paper "**Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework.**"

$$\begin{aligned} &\text{Minimize} \quad J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu \operatorname{Var} x(T) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy } (2.6), \end{cases} \end{aligned} \tag{2.11}$$

where the parameter (representing the weight) $\mu > 0$.

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)] u_i(t) \right\} dt \\ \qquad\qquad + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{2.6}$$

Problem is Not Suitable for Dynamic Programming

- ✓ Fundamental Obstacle
  - The cost function contains , which is nonseparable in the sense of dy-namic programming.
  - More generally, the term can be written as , where is a nonlinear utility function.

- ✓ Why Does Not Fit the Dynamic Programming Framework
  - Dynamic programming relies on the "smoothing property":

  where and .

  - However, this does not hold for :

$P(\mu)$

$$\text{Minimize} \quad J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu \operatorname{Var} x(T)$$

$$\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6),} \end{cases}$$

$$\bar{\lambda} = 1 + 2\mu E\bar{x}(T),$$

$A(\mu, \lambda)$

$$\text{Minimize} \quad E\{\mu x(T)^2 - \lambda x(T)\}$$

$$\text{subject to} \quad \begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)] u_i(t) \right\} dt \\ \qquad\qquad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$

$A(\mu, \lambda)$

Minimize $\quad E\{\mu x(T)^2 - \lambda x(T)\}$

subject to
$$\begin{cases} dx(t) = \left\{r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]\, u_i(t)\right\} dt \\ \qquad\qquad + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$

LQ framework

Minimize $\quad E[\frac{1}{2}\mu y(T)^2]$

subject to
$$\begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\}\, dt \\ \qquad\qquad + \sum_{j=1}^{m} D_j(t)u(t)\, dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases}$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \dots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \dots, \sigma_{mj}(t)). \end{cases}$$

$\boldsymbol{P}(\boldsymbol{\mu})$

$$\begin{array}{ll} \text{Minimize} & J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu \operatorname{Var} x(T) \\ \text{subject to} & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6)}, \end{cases} \end{array}$$

Hence the optimal control for problem P is given by

$$\bar{u}(t, x) = [\sigma(t)\sigma(t)']^{-1} B(t)'(\gamma e^{-\int_t^T r(s)ds} - x).$$

with and given by

$$\bar{\lambda} = e^{\int_0^T \rho(t)dt} + 2\mu x_0 e^{\int_0^T r(t)dt}.$$

Contents

Notation

	fixed time horizon
	Filtered probability space
	Standard Brownian motion
	Natural filtration of augmented by -null sets of
	Action space (non-empty, closed and convex sub-set of)
	Class of measurable, -adapted and square inte-grable processes .

State Process (Mean-Field SDE)

For any $u \in \mathcal{U}$, we consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, \underbrace{\mathbb{E}\psi(x_t)}_{\text{Mean-field term}}, u_t)\mathrm{d}t + \sigma(t, x_t, \underbrace{\mathbb{E}\phi(x_t)}_{\text{Mean-field term}}, u_t)\mathrm{d}B_t, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $b : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}, \quad \psi : \mathbb{R} \longrightarrow \mathbb{R},$

$\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}, \quad \phi : \mathbb{R} \longrightarrow \mathbb{R}.$

The SDE is called mean-field, since and depend not only on , , but also on and .

Cost Functional (Mean-Field Objective)

The expected cost is given by

$$J(u) = \mathbb{E}\left(\int_0^T h(t, x_t, \underbrace{\mathbb{E}\varphi(x_t)}_{\text{Mean-field term}}, u_t)\mathrm{d}t + g(x_T, \underbrace{\mathbb{E}\chi(x_T)}_{\text{Mean-field term}})\right), \tag{2.2}$$

where

$$g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R},$$

$$h : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R},$$

$$\chi : \mathbb{R} \longrightarrow \mathbb{R},$$

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}.$$

Assumptions

(A.1) ψ, ϕ, χ and φ are continuously differentiable. g is continuously differentiable with respect to (x, y). b, σ, h are continuously differentiable with respect to (x, y, v).

(A.2) All the derivatives in (A.1) are Lipschitz continuous and bounded.

$$J(u) = \mathbb{E}\left(\int_0^T h(t, x_t, \mathbb{E}\varphi(x_t), u_t)\mathrm{d}t + g(x_T, \mathbb{E}\chi(x_T))\right),$$

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, \mathbb{E}\psi(x_t), u_t)\mathrm{d}t + \sigma(t, x_t, \mathbb{E}\phi(x_t), u_t)\mathrm{d}B_t, \\ x(0) = x_0, \end{cases}$$

Since and are all Lipschitz continuous it holds that

$$\left| b\left(\cdot,\cdot,\int \phi(x)\mathrm{d}\mu_X(x),\cdot\right) - b\left(\cdot,\cdot,\int \phi(y)\mathrm{d}\mu_Y(y),\cdot\right)\right|$$

$$\leq K\left|\int \phi(x)\mathrm{d}\left(\mu_X(x)-\mu_Y(x)\right)\right|$$

$$\leq K\mathrm{d}\left(\mu_X,\mu_Y\right),$$

where

$$\mathrm{d}(\mu,\nu) = \inf\left\{\left(\mathbb{E}^Q|X-Y|^2\right)^{1/2};\ Q\in\mathcal{P} \text{ with marginals } \mu \text{ and } \nu\right\}$$

Kantorovich-Rubinstein theorem

$$= \sup\left\{\int h\mathrm{d}(\mu-\nu); |h(x)-h(y)|\leq|x-y|\right\},$$

and is the space of probability measures on .

Contents

Notation

	State variable
	Expected value
	Control variable
	Derivative of w.r.t. the state trajectory, the expected value, and the control variable, respectively
	Optimal trajectory and control, respectively

$$\hat{b}(t) = b\left(t, \hat{x}_t, \mathbb{E}\hat{\psi}(t), \hat{u}_t\right)$$

Objective of this Section

This section shows that a necessary condition for optimality is that the optimal control satisfies the Hamiltonian's first-order optimality condition, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}v} H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t\right)\left(v - \hat{u}_t\right) \geq 0$$

We introduce a small variation of optimal control:

where and is a small scalar.

We let denote the state trajectory corresponding to

Then satisfies the following SDE

$$\begin{cases} dx_t^\theta = b(t, x_t^\theta, E[\psi(x_t^\theta)], u_t^\theta)\, dt + \sigma(t, x_t^\theta, E[\phi(x_t^\theta)], u_t^\theta)\, dB_t, \\ x_0^\theta = x_0. \end{cases}$$

Using the Taylor expansion of the perturbed state process around , we obtain the following Lemma.

Lemma 3.1 *Let*

$$\begin{cases} \mathrm{d}z_t = (\hat{b}_x(t)z_t + \hat{b}_y(t)\mathbb{E}(\hat{\psi}_x(t)z_t) + \hat{b}_v(t)v_t)\mathrm{d}t \\ \qquad\qquad + (\hat{\sigma}_x z_t + \hat{\sigma}_y(t)\mathbb{E}(\hat{\phi}_x(t)z_t) + \hat{\sigma}_v(t)v_t)\mathrm{d}B_t, \\ z_0 = 0. \end{cases} \tag{3.1}$$

Then, it holds that

$$\lim_{\theta \to 0} \mathbb{E} \left| \frac{x_t^\theta - \hat{x}_t}{\theta} - z_t \right|_T^{*,2} = 0.$$

Lemma 3.2 *The Gateaux derivative of the cost functional J is given by*

$$\frac{\mathrm{d}}{\mathrm{d}\theta} J\left(\hat{u}+\theta v\right)\Big|_{\theta=0} = \mathbb{E}\left(\int_0^T \left(\hat{h}_x(t) z_t + \hat{h}_y(t)\mathbb{E}\left(\hat{\varphi}_x(t) z_t\right) + \hat{h}_v(t) v_t\right) \mathrm{d}t\right)$$

$$+\,\mathbb{E}\left(\hat{g}_x(T) z_T + \hat{g}_y(T)\mathbb{E}\left(\chi_x(T) z_T\right)\right).$$

To express the Gateaux derivative of the cost function in terms of the Hamiltonian, we define the adjoint equation

$$\begin{cases} \mathrm{d}\hat{p}_t = -(\hat{b}_x(t)\hat{p}_t + \hat{\sigma}_x(t)\hat{q}_t + \hat{h}_x(t))\mathrm{d}t + \hat{q}_t\mathrm{d}B_t \\ \qquad\qquad - (\mathbb{E}(\hat{b}_y(t)\hat{p}_t)\hat{\psi}_x(t) + \mathbb{E}(\hat{\sigma}_y\hat{q}_t)\hat{\phi}_x(t) + \mathbb{E}(\hat{h}_y(t))\hat{\varphi}_x(t))\mathrm{d}t, \\ \hat{p}_T = \hat{g}_x(T) + \mathbb{E}\left(\hat{g}_y(T)\right)\hat{\chi}_x(T). \end{cases}$$

and the Hamiltonian

$$\begin{aligned} H(t, x, u, p, q) := h(t, x, \mathbb{E}(\varphi(x)), u) + b(t, x, \mathbb{E}(\psi(x)), u)\, p \\ + \sigma(t, x, \mathbb{E}(\phi(x)), u)\, q. \end{aligned}$$

Corollary 3.1 *The Gateaux derivative of the cost functional can be expressed in terms of the Hamiltonian H in the following way.*

$$\frac{\mathrm{d}}{\mathrm{d}\theta} J\left(\hat{u}+\theta v\right)\Big|_{\theta=0} = \mathbb{E}\left(\int_0^T \left(\hat{h}_v(t)v_t + \hat{p}_t\hat{b}_v(t)v_t + \hat{q}_t\hat{\sigma}_v(t)v_t\right)\mathrm{d}t\right)$$
$$= \mathbb{E}\left(\int_0^T \frac{\mathrm{d}}{\mathrm{d}v} H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t\right) v_t \mathrm{d}t\right).$$

Main Result

Since U is convex, we may choose the perturbation

$$u_t^\theta = \hat{u}_t + \theta\left(v_t - \hat{u}_t\right) \in \mathcal{U},$$

for $0 \leq \theta \leq 1$.

Thus, we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}\theta} J\left(\hat{u} + \theta\left(v - \hat{u}\right)\right)\Big|_{\theta=0} \overset{\text{Cor 3.1}}{=} \mathbb{E}\left(\int_0^T \frac{\mathrm{d}}{\mathrm{d}v} H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t\right)\left(v_t - \hat{u}_t\right)\mathrm{d}t\right) \geq 0.$$

Main Result

Theorem 3.1 *Under assumptions* (A.1)–(A.2), *if* $\hat{u}_t$ *is an optimal control with state trajectory* $\hat{x}_t$, *then there exists a pair* $(\hat{p}_t, \hat{q}_t)$ *of adapted processes which satisfies* (3.7) *and* (3.8), *such that*

$$\frac{\mathrm{d}}{\mathrm{d}v} H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t\right)\left(v - \hat{u}_t\right) \geq 0, \quad \mathbb{P}\text{-}a.s., \textit{ for all } t \in [0, T]. \tag{3.9}$$

Contents

Sufficient Conditions for Optimality

(A.1) ψ, ϕ, χ and φ are continuously differentiable. g is continuously differentiable with respect to (x, y). b, σ, h are continuously differentiable with respect to (x, y, v).

(A.2) All the derivatives in (A.1) are Lipschitz continuous and bounded.

(A.3) The function g is convex in (x, y).

(A.4) The Hamiltonian is convex in (x, y, v).

(A.5) The functions $\psi, \phi, \varphi, \chi$ are convex.

(A.6) The functions b_y, σ_y, h_y and g_y are non-negative.

$$J(u) = \mathbb{E}\left(\int_0^T h(t, x_t, \mathbb{E}\varphi(x_t), u_t)\mathrm{d}t + g(x_T, \mathbb{E}\chi(x_T))\right),$$

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, \mathbb{E}\psi(x_t), u_t)\mathrm{d}t + \sigma(t, x_t, \mathbb{E}\phi(x_t), u_t)\mathrm{d}B_t, \\ x(0) = x_0, \end{cases}$$

Theorem 4.1 *Assume the conditions* (A.1)–(A.6) *are satisfied and let* $\hat{u} \in \mathcal{U}$ *with state trajectory* $\hat{x}_t$ *be given and such that there exist solutions* $\hat{p}_t, \hat{q}_t$ *to the adjoint equation* (3.7). *Then, if*

$$H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t\right) = \inf_{v \in U} H\left(t, \hat{x}_t, v, \hat{p}_t, \hat{q}_t\right), \tag{4.1}$$

for all $t \in [0, T]$, $\mathbb{P}$-*a.s.,* $\hat{u}$ *is an optimal control.*

Contents

- : risk free bank account
- : risky asset

The price processes evolve according to the equations

$$\begin{cases} \mathrm{d}S_t^0 = \rho_t S_t^0 \mathrm{d}t, \\ \mathrm{d}S_t^1 = \alpha_t S_t^1 \mathrm{d}t + \sigma_t S_t^1 \mathrm{d}B_t, \end{cases}$$

where $\alpha_t, \sigma_t, \rho_t$ are bounded deterministic functions.

- : the amount of money invested in the risky asset at time

Under the self-financing assumption (no external cash flows)

$$dx_t = (\rho_t x_t + (\alpha_t - \rho_t) u_t) dt + \sigma_t u_t dB_t, \quad x_0 = x(0).$$

The cost functional, to be minimized, is given by

$$J(u) = \frac{\gamma}{2}\mathrm{Var}(x_T) - \mathbb{E}(x_T).$$

By rewriting this as

$$\boxed{J(u) = \mathbb{E}\left(\frac{\gamma}{2}x_T^2 - x_T\right) - \frac{\gamma}{2}(\mathbb{E}(x_T))^2}$$

we see that this is a cost functional of the form

$$J(u) = \mathbb{E}\left(\int_0^T h(t, x_t, \mathbb{E}\varphi(x_t), u_t)\mathrm{d}t + \boxed{g(x_T, \mathbb{E}\chi(x_T))}\right)$$

We solve it by writing down the Hamiltonian for this system:

$$H(t, x, u, p, q) = (\rho_t x + (\alpha_t - \rho_t) u) p + \sigma_t u q.$$

The adjoint equation becomes

$$\begin{cases} \mathrm{d}p_t = -\rho_t p_t \mathrm{d}t + q_t \mathrm{d}B_t, \\ p_T = \gamma(x_T - \mu_T) - 1, \end{cases}$$

where $\mu_t = \mathbb{E}(x_t)$.

Looking at the terminal condition of , it is reasonable to

try a solution of the form , with

We get the solution candidate for the mean-variance portfolio selection problem as follows:

$$\hat{u}\left(t, \hat{x}_t\right) = \frac{\alpha_t - \rho_t}{\sigma_t^2}\left(x_0 e^{\int_0^t \rho_s \mathrm{d}s} + \frac{1}{\gamma} e^{\int_0^T \Lambda_s \mathrm{d}s - \int_t^T \rho_s \mathrm{d}s} - \hat{x}_t\right)$$

which is identical to the optimal control found in [1].

[1] Zhou, X. Y. and Li, D. (2000). "Continuous-time mean-variance portfolio selection: a stochastic LQ framework," *Applied Mathematics & Optimization*, 42, pp. 19–33.

Thank you