Convergence rate for ensemble-based solutions to optimal control of uncertain dynamical systems

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Carathéodory function

$f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$, for $\Omega \subseteq \mathbb{R}^d$ endowed with the Lebesgue measure, is a Carathéodory function if

- The mapping $x \mapsto f(x, \xi)$ is Lebesgue-measurable for every $\xi \in \mathbb{R}^N$,
- the mapping $\xi \mapsto f(x, \xi)$ is continuous for almost for almost every $x \in \Omega$.

The main merit of Carathéodory function is the following: If $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and $u : \Omega \to \mathbb{R}^N$ is Lebesgue-measurable, then the composition $x \mapsto f(x, u(x))$ is Lebesgue-measurable.

Setting

We consider the optimal control problem

$$\min_{u \in L^2(0,1;\mathbb{R}^m)} \mathbb{E}\left[F(x^u(1,\xi),\xi)\right] + \psi(u), \tag{1}$$

where for each parameter $\xi \in \Xi$ and control $u(\cdot) \in L^2(0,1;\mathbb{R}^m)$, $x^u(\cdot,\xi) = x(\cdot,\xi)$ solves the parameterized affine-control dynamical system

$$\begin{aligned} \dot{x}(t,\xi) = f_0(x(x,\xi),\xi) + f_1(x(x,\xi),\xi)u(t) \quad \text{for a.e. } t \in (0,1), \\ x(0,\xi) = x_0(\xi), \end{aligned} \tag{2}$$

where Ξ is a complete separable metric space equipped with its Borel sigma-algebra, $F : \mathbb{R}^n \times \Xi \to \mathbb{R}$, $f_0 : \mathbb{R}^n \times \Xi \to \mathbb{R}^n$ and $f_1 : \mathbb{R}^n \times \Xi \to \mathbb{R}^{n\times m}$ are Carathéodory mappings, and $x_0 : \Xi \to \mathbb{R}^n$ is measurable. The function $\psi : L^2(0,1;\mathbb{R}^m) \to (-\infty,\infty]$ is proper and strongly convex with parameter $\alpha > 0$. The parameterized initial value problem in (2) models for uncertain right-hand sides and initial values. With some abuse notation, we use ξ to denote elements of Ξ and a random element taking values in Ξ.

Setting

We use the sample average approximation (SAA) approach to approximate the infinite dimensional optimization problem (1). Throughout the text, let $\xi^1, \xi^2, \cdots$ be i.i.d. Ξ-valued random elements defined on a complete probability space such that each ξ^i has the same distribution as ξ. We obtain the SAA problem

$$\min_{u \in L^2(0,1;\mathbb{R}^m)} \frac{1}{N} \sum_{i=1}^{N} F(x^u(1, \xi^i), \xi^i) + \psi(u). \tag{3}$$

The optimization problem (3) is an optimal control problem with an ensemble of N dynamical systems.

We define parameterized integrand

$$T(u,\xi) := F(x^u(1,\xi),\xi). \tag{4}$$

Furthermore, we define $\psi_\alpha(u) := \psi(u) - (\alpha/2)\|u\|_{L^2(0,1;\mathbb{R}^m)^m}$,

$$g(u) := \mathbb{E}\left[T(u,\xi)\right], \quad \text{and} \quad \hat{g}_N(u) := \frac{1}{N}\sum_{i=1}^{N} T(u,\xi^i). \tag{5}$$

Convergence rates of optimal values

We establish nonasymptotic mean convergence rates for the SAA optimal values. Specifically, we show that for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[|\hat{v}_N^* - v^*|\right] \leq \frac{\mathrm{Const}}{\sqrt{N}} \left(1 + \frac{1}{\sqrt{\alpha}}\right), \tag{6}$$

where v^* is the optimal value of (1) and $\hat{v}_N^*$ is that of (3). Moreover, Const is a constant that does not depend on the sample size N nor on the strong convexity parameter α. However, it can depend on other problem data, such as the control's dimension m.

Convergence rates of optimal values

Theorem

If Assumption $1-3$ *hold and* $u_0 \in dom(\psi)$, *then for all* $N \in \mathbb{N}$,

$$\mathbb{E}\left[|\hat{v}_N^* - v^*\right] \leq \frac{\left(\mathbb{E}\left[(T(u_0,\xi) - \mathbb{E}[T(u_0,\xi)])^2\right]\right)^{1/2}}{\sqrt{N}} + \frac{16\sqrt{3}L'_T r_\psi}{\sqrt{N}}\left(1 + \frac{\rho\sqrt{m}Rm}{\alpha}\right)^{1/2}. \tag{7}$$

Convergence rates for criticality measures

We demonstrate nonasymptotic mean convergence rates for a criticality measure for (1) evaluated at SAA critical points: for each critical point $u_N^* \in \mathrm{dom}(\psi)$ of (3), that is, $\hat{\chi}_N(u_N^*) = 0$, we show that for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[\chi(u_N^*)\right] \leq \frac{\mathrm{Const}}{\sqrt{N}} \left(1 + \frac{1}{\sqrt{\alpha}}\right), \tag{8}$$

where Const is as in (6) and the criticality measures χ and $\hat{\chi}_N$ are defined by

$$\chi(u) := \|u - prox_{\psi_\alpha}(u - \nabla g(u) - \alpha u)\|_{L^2(0,1;\mathbb{R}^m)} \tag{9}$$

and

$$\hat{\chi}(u) := \|u - prox_{\psi_\alpha}(u - \nabla \hat{g}(u) - \alpha u)\|_{L^2(0,1;\mathbb{R}^m)}. \tag{10}$$

Convergence rates for criticality measures

Theorem

Let Assumptions $1-3$ *hold. For each* $N \in \mathbb{N}$*, let* $u_N^* \in dom(\psi)$ *be a measurable critical point of the SAA problem* (3). *If* $u_0 \in dom(\psi)$*, then for all* $N \in \mathbb{N}$*,*

$$\mathbb{E}\left[\chi(u_N^*)\right] \leq \frac{\left(\mathbb{E}\|\nabla_u T(u_0,\xi) - \mathbb{E}[\nabla_u T(u_0,\xi)]\|^2_{L^2(0,1;\mathbb{R}^m)}\right)^{1/2}}{\sqrt{N}} + \frac{16\sqrt{3}L'_{\nabla T} r_\psi}{\sqrt{N}}\left(1 + \frac{\rho\sqrt{m}Rm}{\alpha}\right)^{1/2}. \tag{11}$$