

# Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework

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## **Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework\***

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- 1) Problem Formulation
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Suppose there is a market in which  $m + 1$  assets (or securities) are traded continuously. One of the assets is the *bond* whose price process  $P_0(t)$  is subject to the following (deterministic) ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t) dt, & t \in [0, T], \\ P_0(0) = p_0 > 0, \end{cases} \quad (2.1)$$

where  $r(t) > 0$  is the *interest rate* (of the bond). The other  $m$  assets are *stocks* whose price processes  $P_1(t), \dots, P_m(t)$  satisfy the following stochastic differential equation:

$$\begin{cases} dP_i(t) = P_i(t) \left\{ b_i(t) dt + \sum_{j=1}^m \sigma_{ij}(t) dW^j(t) \right\}, & t \in [0, T], \\ P_i(0) = p_i > 0, \end{cases} \quad (2.2)$$

where  $b_i(t) > 0$  is the *appreciation rate*, and  $\sigma_i(t) \equiv (\sigma_{i1}(t), \dots, \sigma_{im}(t))$ :  $[0, T] \rightarrow R^m$  is the *volatility* or the *dispersion* of the stocks.

Define the *covariance matrix*

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} \equiv (\sigma_{ij}(t))_{m \times m}. \quad (2.3)$$

The basic assumption throughout this paper is

$$\sigma(t)\sigma(t)' \geq \delta I, \quad \forall t \in [0, T], \quad (2.4)$$

for some  $\delta > 0$ . This is the so-called *nondegeneracy* condition. We also assume that all the functions are measurable and uniformly bounded in  $t$ .

Consider an investor whose total wealth at time  $t \geq 0$  is denoted by  $x(t)$ . Suppose he/she decides to hold  $N_i(t)$  shares of  $i$ th asset ( $i = 0, 1, \dots, m$ ) at time  $t$ . Then

$$x(t) = \sum_{i=0}^m N_i(t) P_i(t), \quad t \geq 0. \quad (2.5)$$

Then one has

$$\left\{ \begin{aligned} dx(t) &= \sum_{i=0}^m N_i(t) dP_i(t) \\ &= \left\{ r(t)N_0(t)P_0(t) + \sum_{i=1}^m b_i(t)N_i(t)P_i(t) \right\} dt \\ &\quad + \sum_{i=1}^m N_i(t)P_i(t) \sum_{j=1}^m \sigma_{ij}(t) dW_j(t) \\ &= \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]u_i(t) \right\} dt \\ &\quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t) dW^j(t), \\ x(0) &= x_0 > 0, \end{aligned} \right. \quad (2.6)$$

where

$$u_i(t) \equiv N_i(t)P_i(t), \quad i = 0, 1, 2, \dots, m, \quad (2.7)$$

denotes the total market value of the investor's wealth in the  $i$ th bond/stock. We call  $u(t) = (u_1(t), \dots, u_m(t))'$  a *portfolio* of the investor.

**Definition 2.1.** A portfolio  $u(\cdot)$  is said to be *admissible* if  $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$ .

**Definition 2.2.** The mean-variance portfolio optimization problem is denoted as

$$\begin{aligned} &\text{Minimize} \quad (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \text{Var } x(T)) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6).} \end{cases} \end{aligned} \tag{2.9}$$

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]u_i(t) \right\} dt \\ \quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t) dW^j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{2.6}$$



Moreover, an admissible portfolio  $\bar{u}(\cdot)$  is called an *efficient portfolio* of the problem if there exists no admissible portfolio  $u(\cdot)$  such that

$$J_1(u(\cdot)) \leq J_1(\bar{u}(\cdot)), \quad J_2(u(\cdot)) \leq J_2(\bar{u}(\cdot)), \quad (2.10)$$

and at least one of the inequalities holds *strictly*. In this case, we call  $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in R^2$  an *efficient point*. The set of all efficient points is called the *efficient frontier*.

Therefore, the original problem can be solved via the following optimal control problem:

$$\begin{aligned} \text{Minimize} \quad & J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu \text{Var } x(T) \\ \text{subject to} \quad & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6),} \end{cases} \end{aligned} \tag{2.11}$$

where the parameter (representing the weight)  $\mu > 0$ . Denote the above problem by  $P(\mu)$ . Define

$$\Pi_{P(\mu)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } P(\mu)\}. \tag{2.12}$$

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# Problem $P(\mu)$ is Not Suitable for Dynamic Programming

## ✓ Fundamental Obstacle

- The cost function contains  $[EX(T)]^2$ , which is nonseparable in the sense of dynamic programming.
- More generally, the term  $[EX(T)]^2$  can be written as  $U(EX(T))$ , where  $U$  is a nonlinear utility function.

## ✓ Why $[EX(T)]^2$ Does Not Fit the Dynamic Programming Framework

- Dynamic programming relies on the “smoothing property”:

$$E(E(U[x(T)]|\mathcal{F}_m)|\mathcal{F}_n) = E(U[x(T)]|\mathcal{F}_n)$$

where  $\{\mathcal{F}_k, k = 1, 2, \dots\}$  and  $n \leq m$ .

- However, this does not hold for  $U[Ex(T)]$ :

$$E(U[Ex(T)|\mathcal{F}_m]|\mathcal{F}_n) \neq U[Ex(T)|\mathcal{F}_m].$$

We now propose to embed problem  $P(\mu)$  into a tractable auxiliary problem that turns out to be a stochastic LQ problem. To do this, set the following problem:

$$\begin{aligned} \text{Minimize} \quad & J(u(\cdot); \mu, \lambda) \equiv E\{\mu x(T)^2 - \lambda x(T)\} \\ \text{subject to} \quad & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6),} \end{cases} \end{aligned} \tag{3.1}$$

where the parameters  $\mu > 0$  and  $-\infty < \lambda < +\infty$ . We call the above, problem  $A(\mu, \lambda)$ . Define

$$\Pi_{A(\mu, \lambda)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } A(\mu, \lambda)\}. \tag{3.2}$$

The following result shows the relationship between problems  $P(\mu)$  and  $A(\mu, \lambda)$ .

**Theorem 3.1.** *For any  $\mu > 0$ , one has*

$$\Pi_{P(\mu)} \subseteq \bigcup_{-\infty < \lambda < +\infty} \Pi_{A(\mu, \lambda)}. \quad (3.3)$$

*Moreover, if  $\bar{u}(\cdot) \in \Pi_{P(\mu)}$ , then  $\bar{u}(\cdot) \in \Pi_{A(\mu, \bar{\lambda})}$  with  $\bar{\lambda} = 1 + 2\mu E\bar{x}(T)$ , where  $\bar{x}(\cdot)$  is the corresponding wealth trajectory.*

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# General LQ Problem

The system under consideration in this section is governed by the following linear Ito's stochastic differential equation (SDE):

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)] dt + \sum_{j=1}^m D_j(t)u(t) dW^j(t), \\ x(0) = x_0 \in R^n, \end{cases} \quad (4.1)$$

where  $x_0$  is the initial state,  $W(t) \equiv (W^1(t), \dots, W^m(t))'$  is a given  $m$ -dimensional Brownian motion over  $[0, T]$  on a given filtered probability space  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ , and  $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$  is a control.

For each  $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$ , the associated cost is

$$J(u(\cdot)) = E \left\{ \int_0^T \frac{1}{2} [x(t)' Q(t) x(t) + u(t)' R(t) u(t)] dt + \frac{1}{2} x(T)' H x(T) \right\}. \quad (4.2)$$



We introduce the following *stochastic Riccati equation*:

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)'P(t) - Q(t) \\ \quad + P(t)B(t) \left( R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t) \right)^{-1} B(t)'P(t), \\ P(T) = H, \\ K(t) \equiv R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t) > 0, \quad \forall t \in [0, T], \end{cases} \quad (4.3)$$

along with an equation

$$\begin{cases} \dot{g}(t) = -A(t)'g(t) + P(t)B(t) \left( R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t) \right)^{-1} B(t)'g(t) \\ \quad - P(t)f(t), \\ g(T) = 0. \end{cases} \quad (4.4)$$

**Theorem 4.1.** *If (4.3) and (4.4) admit solutions  $P \in C([0, T]; S_+^n)$  and  $g \in C([0, T]; R^n)$ , respectively, then the stochastic LQ problem (4.1)–(4.2) has an optimal feedback control*

$$u^*(t, x) = - \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)' (P(t)x + g(t)). \quad (4.5)$$

*Moreover, the optimal cost value is*

$$\begin{aligned} J^* = & \frac{1}{2} \int_0^T \left( 2f(t)'g(t) - g(t)B(t) \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)'g(t) \right) dt \\ & + \frac{1}{2} x_0' P(0) x_0 + x_0 g(0). \end{aligned} \quad (4.6)$$

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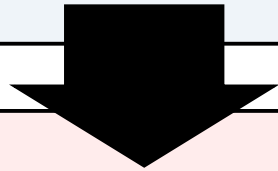
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Set  $\gamma = \frac{\lambda}{2\mu}$  and  $y(t) = x(t) - \gamma$ .

$A(\mu, \lambda)$

Minimize  $E\{\mu x(T)^2 - \lambda x(T)\}$

subject to 
$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]u_i(t) \right\} dt \\ \quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t) dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$



Minimize  $E[\frac{1}{2}\mu y(T)^2]$

subject to 
$$\begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\} dt \\ \quad + \sum_{j=1}^m D_j(t)u(t) dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases}$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \dots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \dots, \sigma_{mj}(t)). \end{cases}$$

$$\text{Minimize} \quad E[\tfrac{1}{2}\mu y(T)^2] \quad (5.2)$$

$$\text{subject to} \quad \begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\} dt \\ \quad \quad \quad + \sum_{j=1}^m D_j(t)u(t) dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases} \quad (5.3)$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \dots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \dots, \sigma_{mj}(t)). \end{cases}$$

Thus, problem (5.2)–(5.3) is a special case of problem (4.1)–(4.2) with

$$(Q(t), R(t)) = (0, 0), \quad H = \mu,$$

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)] dt + \sum_{j=1}^m D_j(t)u(t) dW^j(t), \\ x(0) = x_0 \in R^n, \end{cases} \quad (4.1)$$

$$J(u(\cdot)) = E \left\{ \int_0^T \tfrac{1}{2} [x(t)' Q(t)x(t) + u(t)' R(t)u(t)] dt + \tfrac{1}{2} x(T)' H x(T) \right\}. \quad (4.2)$$

**Theorem 4.1.** *If (4.3) and (4.4) admit solutions  $P \in C([0, T]; S_+^n)$  and  $g \in C([0, T]; R^n)$ , respectively, then the stochastic LQ problem (4.1)–(4.2) has an optimal feedback control*

$$u^*(t, x) = - \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)' (P(t)x + g(t)). \quad (4.5)$$

*Moreover, the optimal cost value is*

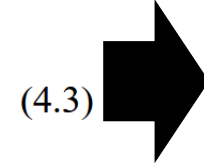
$$\begin{aligned} J^* = & \frac{1}{2} \int_0^T \left( 2f(t)' g(t) - g(t) B(t) \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)' g(t) \right) dt \\ & + \frac{1}{2} x_0' P(0) x_0 + x_0 g(0). \end{aligned} \quad (4.6)$$

Denote

$$\rho(t) = B(t) \left[ \sum_{j=1}^m D_j(t)' D_j(t) \right]^{-1} B(t)' = B(t) [\sigma(t) \sigma(t)']^{-1} B(t)'.$$

We introduce the following *stochastic Riccati equation*:

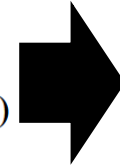
$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)'P(t) - Q(t) \\ \quad + P(t)B(t) \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)' P(t), \\ P(T) = H, \\ K(t) \equiv R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) > 0, \quad \forall t \in [0, T], \end{cases} \quad (4.3)$$



$$\begin{cases} \dot{P}(t) = (\rho(t) - 2r(t)) P(t), \\ P(T) = \mu, \\ P(t) [\sigma(t) \sigma(t)'] > 0, \quad t \in [0, T]. \end{cases} \quad (5.7)$$

along with an equation

$$\begin{cases} \dot{g}(t) = -A(t)' g(t) + P(t) B(t) \left( R(t) + \sum_{j=1}^m D_j(t)' P(t) D_j(t) \right)^{-1} B(t)' g(t) \\ \quad - P(t) f(t), \\ g(T) = 0. \end{cases} \quad (4.4)$$



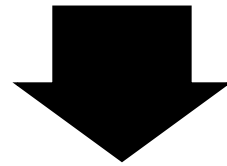
$$\begin{cases} \dot{g}(t) = (\rho(t) - r(t)) g(t) - \gamma r(t) P(t), \\ g(t) = 0, \end{cases} \quad (5.8)$$

Let  $h(t) = g(t)/P(t)$ .

$A(\mu, \lambda)$

Minimize  $E\{\mu x(T)^2 - \lambda x(T)\}$

subject to 
$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]u_i(t) \right\} dt \\ \quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t) dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$



**Solution**

$$\bar{u}(t, x) = [\sigma(t)\sigma(t)']^{-1} B(t)' (\gamma e^{-\int_t^T r(s)ds} - x).$$



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**Definition 2.1.** A portfolio  $u(\cdot)$  is said to be *admissible* if  $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$ .

**Definition 2.2.** The mean-variance portfolio optimization problem is denoted as

$$\begin{aligned} &\text{Minimize} \quad (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \text{Var } x(T)) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6).} \end{cases} \end{aligned} \quad (2.9)$$

Moreover, an admissible portfolio  $\bar{u}(\cdot)$  is called an *efficient portfolio* of the problem if there exists no admissible portfolio  $u(\cdot)$  such that

$$J_1(u(\cdot)) \leq J_1(\bar{u}(\cdot)), \quad J_2(u(\cdot)) \leq J_2(\bar{u}(\cdot)), \quad (2.10)$$

and at least one of the inequalities holds *strictly*. In this case, we call  $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in R^2$  an *efficient point*. The set of all efficient points is called the *efficient frontier*.

**Theorem 6.1.** *The efficient frontier of the bicriteria optimal portfolio selection problem (2.9), if it ever exists, must be given by (6.9).*

$$\begin{aligned} &\text{Minimize} \quad (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \text{Var } x(T)) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6).} \end{cases} \end{aligned} \quad (2.9)$$

Efficient frontier of (2.9)

$$\text{Var } \bar{x}(T) = \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}} (E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt})^2. \quad (6.9)$$

$$\text{Var } \bar{x}(T) = \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}} (E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt})^2. \quad (6.9)$$

Capital Market Line

$\sigma_{\bar{x}(T)}$ : standard deviation of the terminal wealth

$$E\bar{x}(T) = x_0 e^{\int_0^T r(t)dt} + \sqrt{\frac{1 - e^{-\int_0^T \rho(t)dt}}{e^{-\int_0^T \rho(t)dt}}} \sigma_{\bar{x}(T)}. \quad (6.10)$$

Price of Risk

Thank you