Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework

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Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework*

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1) Problem Formulation

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Suppose there is a market in which $m+1$ assets (or securities) are traded continuously. One of the assets is the *bond* whose price process $P_0(t)$ is subject to the following (deterministic) ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t)\,dt, & t \in [0, T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{2.1}$$

where $r(t) > 0$ is the *interest rate* (of the bond). The other m assets are *stocks* whose price processes $P_1(t), \ldots, P_m(t)$ satisfy the following stochastic differential equation:

$$\begin{cases} dP_i(t) = P_i(t)\left\{b_i(t)\,dt + \sum_{j=1}^{m} \sigma_{ij}(t)\,dW^j(t)\right\}, & t \in [0, T], \\ P_i(0) = p_i > 0, \end{cases} \tag{2.2}$$

where $b_i(t) > 0$ is the *appreciation rate*, and $\sigma_i(t) \equiv (\sigma_{i1}(t), \ldots, \sigma_{im}(t))\colon [0, T] \to R^m$ is the *volatility* or the *dispersion* of the stocks.

Define the *covariance matrix*

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} \equiv (\sigma_{ij}(t))_{m\times m}. \tag{2.3}$$

The basic assumption throughout this paper is

$$\sigma(t)\sigma(t)' \geq \delta I, \qquad \forall t \in [0, T], \tag{2.4}$$

for some $\delta > 0$. This is the so-called *nondegeneracy* condition. We also assume that all the functions are measurable and uniformly bounded in t.

Consider an investor whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Suppose he/she decides to hold $N_i(t)$ shares of ith asset ($i = 0, 1, \ldots, m$) at time t. Then

$$x(t) = \sum_{i=0}^{m} N_i(t) P_i(t), \qquad t \geq 0. \tag{2.5}$$

Then one has

$$\begin{cases} dx(t) = \sum_{i=0}^m N_i(t)\, dP_i(t) \\ \qquad = \left\{ r(t) N_0(t) P_0(t) + \sum_{i=1}^m b_i(t) N_i(t) P_i(t) \right\} dt \\ \qquad\quad + \sum_{i=1}^m N_i(t) P_i(t) \sum_{j=1}^m \sigma_{ij}(t)\, dW_j(t) \\ \qquad = \left\{ r(t) x(t) + \sum_{i=1}^m [b_i(t) - r(t)]\, u_i(t) \right\} dt \\ \qquad\quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{2.6}$$

where

$$u_i(t) \equiv N_i(t) P_i(t), \qquad i = 0, 1, 2, \ldots, m, \tag{2.7}$$

denotes the total market value of the investor's wealth in the ith bond/stock. We call $u(t) = (u_1(t), \ldots, u_m(t))'$ a *portfolio* of the investor.

Definition 2.1. A portfolio $u(\cdot)$ is said to be *admissible* if $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$.

Definition 2.2. The mean-variance portfolio optimization problem is denoted as

$$\begin{aligned} &\text{Minimize} \quad (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \operatorname{Var} x(T)) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6).} \end{cases} \end{aligned} \tag{2.9}$$

$$\begin{cases} dx(t) = \left\{r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)] u_i(t)\right\} dt \\ \qquad\qquad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{2.6}$$

Moreover, an admissible portfolio $\bar{u}(\cdot)$ is called an *efficient portfolio* of the problem if there exists no admissible portfolio $u(\cdot)$ such that

$$J_1(u(\cdot)) \leq J_1(\bar{u}(\cdot)), \qquad J_2(u(\cdot)) \leq J_2(\bar{u}(\cdot)), \tag{2.10}$$

and at least one of the inequalities holds *strictly*. In this case, we call $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in R^2$ an *efficient point*. The set of all efficient points is called the *efficient frontier*.

Therefore, the original problem can be solved via the following optimal control problem:

$$\begin{aligned} &\text{Minimize} \quad J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu \operatorname{Var} x(T) \\ &\text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6)}, \end{cases} \end{aligned} \tag{2.11}$$

where the parameter (representing the weight) $\mu > 0$. Denote the above problem by $P(\mu)$. Define

$$\Pi_{P(\mu)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } P(\mu)\} . \tag{2.12}$$

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Problem $P(\mu)$ is Not Suitable for Dynamic Programming

✓ Fundamental Obstacle

- The cost function contains $[EX(T)]^2$, which is nonseparable in the sense of dynamic programming.
- More generally, the term $[EX(T)]^2$ can be written as $U(EX(T))$, where U is a nonlinear utility function.

✓ Why $[EX(T)]^2$ Does Not Fit the Dynamic Programming Framework

- Dynamic programming relies on the “smoothing property”:

$$E(E(U[x(T)]|\mathcal{F}_m)|\mathcal{F}_n) = E(U[x(T)]|\mathcal{F}_n)$$

where $\{\mathcal{F}_k, k = 1,2, \dots\}$ and $n \leq m$.

- However, this does not hold for $U[Ex(T)]$:

$$E(U[Ex(T)|\mathcal{F}_m]|\mathcal{F}_n) \neq U[Ex(T)|\mathcal{F}_m].$$

We now propose to embed problem $P(\mu)$ into a tractable auxiliary problem that turns out to be a stochastic LQ problem. To do this, set the following problem:

$$
\begin{array}{ll}
\text{Minimize} & J(u(\cdot);\mu,\lambda) \equiv E\{\mu x(T)^2 - \lambda x(T)\} \\
\text{subject to} & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0,T;R^m), \\ (x(\cdot),u(\cdot)) \text{ satisfy (2.6)}, \end{cases}
\end{array} \tag{3.1}
$$

where the parameters $\mu > 0$ and $-\infty < \lambda < +\infty$. We call the above, problem $A(\mu, \lambda)$. Define

$$
\Pi_{A(\mu,\lambda)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } A(\mu,\lambda)\}. \tag{3.2}
$$

The following result shows the relationship between problems $P(\mu)$ and $A(\mu, \lambda)$.

Theorem 3.1. *For any $\mu > 0$, one has*

$$\Pi_{P(\mu)} \subseteq \bigcup_{-\infty<\lambda<+\infty} \Pi_{A(\mu,\lambda)}. \tag{3.3}$$

Moreover, if $\bar{u}(\cdot) \in \Pi_{P(\mu)}$, then $\bar{u}(\cdot) \in \Pi_{A(\mu,\bar{\lambda})}$ with $\bar{\lambda} = 1 + 2\mu E\bar{x}(T)$, where $\bar{x}(\cdot)$ is the corresponding wealth trajectory.

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General LQ Problem

The system under consideration in this section is governed by the following linear Ito's stochastic differential equation (SDE):

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]\,dt + \sum_{j=1}^{m} D_j(t)u(t)\,dW^j(t), \\ x(0) = x_0 \in R^n, \end{cases} \tag{4.1}$$

where x_0 is the initial state, $W(t) \equiv (W^1(t), \ldots, W^m(t))'$ is a given m-dimensional Brownian motion over $[0, T]$ on a given filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$, and $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$ is a control.

For each $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$, the associated cost is

$$J(u(\cdot)) = E\left\{\int_0^T \tfrac{1}{2}[x(t)'Q(t)x(t) + u(t)'R(t)u(t)]\,dt + \tfrac{1}{2}x(T)'Hx(T)\right\}. \tag{4.2}$$

We introduce the following *stochastic Riccati equation*:

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)'P(t) - Q(t) \\ \qquad\qquad + P(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'P(t), \\ P(T) = H, \\ K(t) \equiv R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t) > 0, \qquad \forall t \in [0, T], \end{cases} \tag{4.3}$$

along with an equation

$$\begin{cases} \dot{g}(t) = -A(t)'g(t)+P(t)B(t)\left(R(t)+\sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'g(t) \\ \qquad\qquad -P(t)f(t), \\ g(T) = 0. \end{cases} \tag{4.4}$$

Theorem 4.1. *If* (4.3) *and* (4.4) *admit solutions* $P \in C([0,T]; S^n_+)$ *and* $g \in C([0,T]; R^n)$, *respectively, then the stochastic LQ problem* (4.1)–(4.2) *has an optimal feedback control*

$$u^*(t,x) = -\left(R(t) + \sum_{j=1}^{m} D_j(t)' P(t) D_j(t)\right)^{-1} B(t)'(P(t)x + g(t)). \tag{4.5}$$

Moreover, the optimal cost value is

$$J^* = \tfrac{1}{2}\int_0^T \left(2f(t)'g(t) - g(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)' P(t) D_j(t)\right)^{-1} B(t)'g(t)\right) dt$$
$$+ \tfrac{1}{2}x_0' P(0) x_0 + x_0 g(0). \tag{4.6}$$

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Set $\gamma = \dfrac{\lambda}{2\mu}$ and $y(t) = x(t) - \gamma$.

$A(\mu, \lambda)$

Minimize $E\{\mu x(T)^2 - \lambda x(T)\}$

subject to
$$\begin{cases} dx(t) = \left\{r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]\, u_i(t)\right\} dt \\ \qquad + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$

Minimize $E[\frac{1}{2}\mu y(T)^2]$

subject to
$$\begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\}\, dt \\ \qquad + \sum_{j=1}^{m} D_j(t)u(t)\, dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases}$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \ldots, \sigma_{mj}(t)). \end{cases}$$

$$\text{Minimize} \quad E[\tfrac{1}{2}\mu y(T)^2] \tag{5.2}$$

$$\text{subject to} \quad \begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\}\,dt \\ \qquad\qquad + \sum_{j=1}^{m} D_j(t)u(t)\,dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases} \tag{5.3}$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \ldots, \sigma_{mj}(t)). \end{cases}$$

Thus, problem (5.2)–(5.3) is a special case of problem (4.1)–(4.2) with

$$(Q(t), R(t)) = (0, 0), \qquad H = \mu,$$

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]\,dt + \sum_{j=1}^{m} D_j(t)u(t)\,dW^j(t), \\ x(0) = x_0 \in R^n, \end{cases} \tag{4.1}$$

$$J(u(\cdot)) = E\left\{\int_0^T \tfrac{1}{2}[x(t)'Q(t)x(t) + u(t)'R(t)u(t)]\,dt + \tfrac{1}{2}x(T)'Hx(T)\right\}. \tag{4.2}$$

Theorem 4.1. *If* (4.3) *and* (4.4) *admit solutions* $P \in C([0, T]; S^n_+)$ *and* $g \in C([0, T]; R^n)$, *respectively, then the stochastic LQ problem* (4.1)–(4.2) *has an optimal feedback control*

$$u^*(t, x) = -\left(R(t) + \sum_{j=1}^{m} D_j(t)' P(t) D_j(t)\right)^{-1} B(t)'(P(t)x + g(t)). \tag{4.5}$$

Moreover, the optimal cost value is

$$\begin{aligned} J^* = \tfrac{1}{2}\int_0^T &\left(2f(t)'g(t) - g(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'g(t)\right) dt \\ &+ \tfrac{1}{2}x_0' P(0)x_0 + x_0 g(0). \end{aligned} \tag{4.6}$$

Denote

$$\rho(t) = B(t)\left[\sum_{j=1}^{m} D_j(t)'D_j(t)\right]^{-1} B(t)' = B(t)[\sigma(t)\sigma(t)']^{-1}B(t)'.$$

We introduce the following *stochastic Riccati equation*:

$$\begin{cases} \dot P(t) = -P(t)A(t) - A(t)'P(t) - Q(t) \\ \qquad + P(t)B(t)\left(R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'P(t), \\ P(T) = H, \\ K(t) \equiv R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t) > 0, \qquad \forall t \in [0,T], \end{cases} \tag{4.3}$$

along with an equation

$$\begin{cases} \dot g(t) = -A(t)'g(t) + P(t)B(t)\left(R(t)+\sum_{j=1}^m D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'g(t) \\ \qquad -P(t)f(t), \\ g(T) = 0. \end{cases} \tag{4.4}$$

$$\begin{cases} \dot P(t) = (\rho(t) - 2r(t))\,P(t), \\ P(T) = \mu, \\ P(t)[\sigma(t)\sigma(t)'] > 0, \qquad t \in [0,T]. \end{cases} \tag{5.7}$$

$$\begin{cases} \dot g(t) = (\rho(t) - r(t))\,g(t) - \gamma r(t)P(t), \\ g(t) = 0, \end{cases} \tag{5.8}$$

Let $h(t) = g(t)/P(t)$.

$\boldsymbol{A(\mu, \lambda)}$

Minimize $\quad E\{\mu x(T)^2 - \lambda x(T)\}$

subject to

$$\begin{cases} dx(t) = \left\{r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]\, u_i(t)\right\} dt \\ \qquad\qquad + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t) u_i(t)\, dW^j(t), \\ x(0) = x_0 > 0 \end{cases}$$

Solution

$$\bar{u}(t, x) = [\sigma(t)\sigma(t)']^{-1} B(t)'(\gamma e^{-\int_t^T r(s)ds} - x).$$

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Definition 2.2. The mean-variance portfolio optimization problem is denoted as

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Moreover, an admissible portfolio $\bar{u}(\cdot)$ is called an *efficient portfolio* of the problem if there exists no admissible portfolio $u(\cdot)$ such that

$$J_1(u(\cdot)) \leq J_1(\bar{u}(\cdot)), \qquad J_2(u(\cdot)) \leq J_2(\bar{u}(\cdot)), \tag{2.10}$$

and at least one of the inequalities holds *strictly*. In this case, we call $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in R^2$ an *efficient point*. The set of all efficient points is called the *efficient frontier*.

Theorem 6.1. *The efficient frontier of the bicriteria optimal portfolio selection problem* (2.9), *if it ever exists, must be given by* (6.9).

$$\begin{aligned} &\text{Minimize} \quad (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \operatorname{Var} x(T)) \\ &\quad \text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy } (2.6). \end{cases} \end{aligned} \tag{2.9}$$

Efficient frontier of (2.9)

$$\operatorname{Var} \bar{x}(T) = \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}} \left(E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt}\right)^2. \tag{6.9}$$

$$\mathrm{Var}\,\bar{x}(T) = \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}} \left(E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt}\right)^2. \qquad (6.9)$$

$\sigma_{\bar{x}(T)}$: standard deviation of the terminal wealth

Capital Market Line

$$E\bar{x}(T) = x_0 e^{\int_0^T r(t)dt} + \sqrt{\frac{1 - e^{-\int_0^T \rho(t)dt}}{e^{-\int_0^T \rho(t)dt}}}\,\sigma_{\bar{x}(T)}. \qquad (6.10)$$

Price of Risk

Thank you