Stochastic Differential Equations

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Overview

Definition

If Ω is a given set, then σ-algebra $\mathcal{F}$ on Ω is a family of subsets of Ω with the following properties:

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition

A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \Rightarrow [0, 1]$ such that

1. $P(\emptyset) = 0, \; P(\Omega) = 1$
2. If $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{1}$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.
It is called complete probability space if $\mathcal{F}$ contains all subsets of G of Ω with P-outer measure zero, i.e. with

$$P^*(G) := \inf\{P(F); F \in \mathcal{F}, G \subset F\} = 0. \tag{2}$$

Some Mathematical Preliminaries

Definition

If $(\Omega, \mathcal{F}, P)$ is given probability space, then a function $Y : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F} \tag{3}$$

for all open sets $U \in \mathbb{R}^n$. The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets.

Some Mathematical Preliminaries

Definition

Given any family $\mathcal{U}$ of subsets of Ω there is a smallest σ-algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \bigcap \{\mathcal{H}; \mathcal{H} \ \sigma\text{-algebra of } \Omega, \ \mathcal{U} \subset \mathcal{H}\}. \tag{4}$$

We call $\mathcal{H}_{\mathcal{U}}$ the σ-algebra generated by $\mathcal{U}$.

Definition

If $\mathcal{U}$ is the collection of all open subsets of a topological space Ω, then $\mathcal{B} = \mathcal{H}_{\mathcal{U}}$ is called the Borel σ-algebra on Ω and the elements $B \in \mathcal{B}$ are called Borel sets.

Definition

A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t. \tag{5}$$

Definition

A stochastic process is a parameterized collection of random variables

$$\{X_t\}_{t \in T} \tag{6}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

The parameter space T is usually the halfline $[0, \infty)$.

Definition

Note that for each $t \in T$ fixed, we have a random variable

$$\omega \to X_t(\omega); \quad \omega \in \Omega. \tag{7}$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \to X_t(\omega); \quad t \in T \tag{8}$$

which is called a path of X_t.

Consider the simple population growth model

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0 \tag{9}$$

where $N(t)$ is the size of the population at time t, and $a(t)$ is the relative rate of growth at time t. It might happen that $a(t)$ is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + \text{“noise”}. \tag{10}$$

Construction of the Ito Integral

Therefore, (9) becomes

$$\frac{dN}{dt} = r(t)N(t) + N(t)\,\text{``noise''}. \tag{11}$$

That is, in form of integration,

$$N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t N(s)\,\text{``noise''}\,ds. \tag{12}$$

Questions

What is the mathematical interpretation for the "noise" term.

It turns out that a reasonable mathematical interpretation for the "noise" term is the white noise $\dot{B}(t)$, which is formally regarded as the derivative of a Brownian motion $B(t)$.
Using that $\dot{B}(t)dt = dB(t)$, we rewrite (12) as follows

$$N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t N(s)dB_s. \tag{13}$$

Construction of Ito Integral

More generally in equations of the form

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)\text{“noise”}, \tag{14}$$

where b and σ are some given functions, we obtain

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{15}$$

Question

We have to show that the following term is integrable

$$\int_S^T f(t, \omega)dB_t(\omega). \tag{16}$$

Brownian Motion

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A Brownian motion is function satisfying the following conditions:

1. $B_0 = 0$;
2. (stationary increments) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;
3. (independent increments) the random variables $B_{t_i} - B_{s_i}$ are mutually independent if the intervals $[s_i, t_i]$ are nonoverlapping.

Brownian motion is nowhere differentiable.

We want to define the Ito integral

$$\mathcal{I}[f](\omega) = \int_S^T f(t,\omega)dB_t(\omega). \tag{17}$$

Idea

1. We show that each f can be approximated by a simple class of functions ϕ's.
2. We define $\int fdB$ as the limit of $\int \phi dB$ as $\phi \to f$.

The Ito isometry

Lemma

If $\phi(t,\omega)$ *is bounded and elementary then*

$$\mathbb{E}\left[\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T \phi(t,\omega)^2 dt\right]. \tag{18}$$

The following Step 1-3 are to use the previous isometry to extend the definition from elementary functions to functions in $\mathcal{V}$. We do this in 3 steps:

Step 1

Let $g \in \mathcal{V}$ be bounded and $g(\cdot, \omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathcal{V}$ such that

$$\mathbb{E}\left[\int_S^T (g - \phi_n)^2 dt\right] \to 0 \quad \text{as} \quad n \to \infty. \tag{19}$$

Step 2

Let $h \in \mathcal{V}$ be bounded. Then there exist bounded functions $g_n \in \mathcal{V}$ such that $g_n(\cdots, \omega)$ is continuous for all ω and n, and

$$\mathbb{E}\left[\int_S^T (h - g_n)^2 dt\right] \to 0. \tag{20}$$

The Ito Integral

Step 3

Let $f \in \mathcal{V}$. Then there exists a sequence $\{h_n\} \subset \mathcal{V}$ such that h_n is bounded for each n and

$$\mathbb{E}\left[\int_S^T (f - h_n)^2 dt\right] \to 0 \quad \text{as} \quad n \to \infty. \tag{21}$$

If $f \in \mathcal{V}$, we choose, by Steps 1-3, elementary functions $\phi_n \in \mathcal{V}$ such that

$$\mathbb{E}\left[\int_S^T (f - \phi_n)^2 dt\right] \to 0. \tag{22}$$

The Ito Integral

Definition

Let $f \in \mathcal{V}(S,T)$. Then the Ito integral of f is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega) \tag{23}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$\mathbb{E}\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2\,dt\right] \to 0 \quad \text{as} \quad n \to \infty \tag{24}$$

From the previous definition, we get the following important

Corollary

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega)dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(t,\omega)dt\right]. \tag{25}$$

Ito Integral

An important property of the Ito integral is that it is a martingale.

Definition

An n-dimensional stochastic process $\{\mathcal{M}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ if

1. M_t is $\mathcal{M}_t$-measurable for all t,
2. $\mathbb{E}[|M_t|] < \infty$ for all t,
3. $\mathbb{E}[M_s|\mathcal{M}_t] = M_t$ for all $s \geq t$.

Martingale

Definition

A discrete-time stochastic process $\{X_0, X_1, \cdots\}$ is a martingale if

$$X_t = \mathbb{E}\left[X_{t+1} | \mathcal{F}_t\right], \tag{26}$$

for all $t \geq 0$, wehre $\mathcal{F}_t = \{X_0, \cdots, X_t\}$.

Proposition

For all $t \geq s$, we have $X_s = \mathbb{E}[X_t | \mathcal{F}_s]$.

Quadratic Variation

Theorem

For a partition $\Pi = \{t_0, t_1, \cdots, t_N\}$ *of an interval* $[0, T]$*, let* $|\Pi| = \max_i(t_{i+1} - t_i)$*. A Brownian motion* B_t *satisfies the following equation with probability 1:*

$$\lim_{|\Pi|\to 0} \sum_i \left(B_{t_{i+1}} - B_{t_i}\right)^2 = T \tag{27}$$

The above can be summarized by the differential equation $(dB_t)^2 = dt$.

Ito Lemma

Multiplication Rule

- $(dB_t)^2 = dt$;
- $(dt)^2 = 0$;
- $dt \cdot dB_t = dB_t \cdot dt = 0$.

Ito Lemma

Simple Ito Lemma

$$df = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \tag{28}$$

Ito Lemma

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t \tag{29}$$

Ito Lemma

Theorem

Let $f(t,x)$ be a smooth function of two variables, and let X_t be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t dB_t$ for a Brownian motion B_t. Then

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dB_t. \tag{30}$$

References

- Stochastic Differential Equations and their Applications
- Stochastic Differential Equations
- MIT 18.S096 Topics in Mathematics w Applications in Finance

Thank you for listening